

Box Resolvability

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Abstract. We say that a topological group G is partially box κ -resolvable if there exist a dense subset B of G and a subset A of G , $|A| = \kappa$ such that the subsets $\{aB : a \in A\}$ are pairwise disjoint. If $G = AB$ then G is called box κ -resolvable. We prove two theorems. If a topological group G contains an injective convergent sequence then G is box ω -resolvable. Every infinite totally bounded topological group G is partially box n -resolvable for each natural number n , and G is box κ -resolvable for each infinite cardinal κ , $\kappa < |G|$.

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1. Introduction

For a cardinal κ , a topological space X is called κ -*resolvable* if X can be partitioned into κ dense subsets [1]. In the case $\kappa = 2$, these spaces were defined by Hewitt [6] as *resolvable spaces*. If X is not κ -resolvable then X is called κ -*irresolvable*.

In topological groups, the intensive study of resolvability was initiated by the following remarkable theorem of Comfort and van Mill [3]: every countable non-discrete Abelian topological group G with finite subgroup $B(G)$ of elements of order 2 is 2-resolvable. In fact [19], every infinite Abelian group G with finite $B(G)$ can be partitioned into ω subsets dense in every non-discrete group topology on G . On the other hand, under MA, the countable Boolean group G , $G = B(G)$ admits maximal (hence, 2-irresolvable) group topology [8]. Every non-discrete ω -irresolvable topological group G contains an open countable Boolean subgroup provided that G is Abelian [11] or countable [18], but the existence of non-discrete ω -irresolvable group topology on the countable Boolean group implies that there is a P -point in ω^* [11]. Thus, in some models of ZFC (see [14]), every non-discrete Abelian or countable topological group is ω -resolvable. We mention also κ -resolvability of every infinite totally bounded topological group G of cardinality κ [9]. For systematic exposition of resolvability in topological and left topological group see [4, Chapter 13].

This note is to introduce more delicate kind of resolvability, the box resolvability.

Given a group G and a cardinal κ , we say that a subset B of G is a *partial box* of index κ , if there exists a subset A of G , $|A| = \kappa$ such that the subsets $\{aB : a \in A\}$ are pairwise disjoint. In addition, if $G = AB$ then B is called a *box* of index κ . Example: a subgroup H of G is a box of index $|G : H|$, and any set R of representatives of right cosets of G by H is a box of index $|H|$.

We use also the factorization terminology [16]. For subset A, B of G , the product AB is called a *partial factorization* if $aB \cap a'B = \emptyset$ for any distinct $a, a' \in A$. If $G = AB$ then the product AB is called a *factorization* of G . Thus, a box B of index κ is a right factor of some factorization $G = AB$ such that $|A| = \kappa$.

We say that a topological group G is (*partially*) *box* κ -resolvable if there exists a (partial)

box B of index κ dense in G . Clearly, every partially box κ -resolvable topological group is κ -resolvable, but a κ -resolvable group needs not to be box κ -resolvable (see Examples 1 and 2). However, I do not know, whether every 2-resolvable group is partially box 2-resolvable.

On exposition: in section 2, we prove two theorems announced in Abstract and discuss some prospects of box resolvability in section 3.

2. Results

We begin with two examples demonstrating purely algebraic obstacles to finite box resolvability.

Example 1. We assume that the group \mathbb{Z} of integer numbers is factorized $\mathbb{Z} = A + B$ so that A is finite, $|A| > 1$. By the Hajós theorem [5], B is periodic: $B = m + B$ for some $m \neq 0$. Then $m\mathbb{Z} + B = B$ and $m\mathbb{Z} + b \subseteq B$ for $b \in B$.

Now we endow \mathbb{Z} with the topology τ of finite indices (having $\{n\mathbb{Z} : z \in \mathbb{N}\}$ as the base at 0). Since $m\mathbb{Z}$ is open in τ and $|A| > 1$, we conclude that B is not dense, so (\mathbb{Z}, τ) is box n -irresolvable for each $n > 1$. \square

Example 2. Every torsion group G without elements of order 2 has no boxes of index 2. We assume the contrary: $G = B \cup gB$, $B \cap gB = \emptyset$ and $e \in B$, e is the identity of G . Then $g^2B = B$ and B contains the subgroup $\langle g^2 \rangle$ generated by g . Since g is an element of odd order, we have $g \in \langle g^2 \rangle$ and $g \in B \cap gB$. \square

Let G be a countable group. Applying [13, Theorem 2], we can find a factorization $G = AB$ such that $|A| = |B| = \omega$. Hence, if we endow G with a group topology τ , there are no algebraic obstacles to box ω -resolvability of (G, τ) .

In what follows, we use two elementary observations. Let G be a topological group, H be a subgroup G and R be a system of representatives of right cosets of G by H . Let AB be a factorization of H . Then we have

- (1) If B is dense in A then $A(BR)$ is a factorization of G with dense BR ;
- (2) If R is dense in G then $A(BR)$ is a factorization of G with dense BR .

Example 3. Let G be a non-discrete metrizable group and let A be a subgroup of G . If A is either finite or countable discrete then there is a factorization AB of G such that B is dense in G .

In view of (1), we may suppose that G is countable. Let $\{U_n : n \in \omega\}$ be a base of topology of G . For each $n \in \omega$, we choose $x_n \in U_n$ so that $Ax_n \cap Ax_m = \emptyset$ if $n \neq m$. Then we complement the set $\{x_n : n \in \omega\}$ to some full system B of representatives of right cosets of G by A . \square

If a topological group G contains an injective convergent sequence then G is ω -resolvable (see [2, Lemma 5.4]). If an injective sequence $(a_n)_{n \in \omega}$ converges to the identity e in some group topology on a group G then, by [13, Theorem 1], the set $\{e, a_n, a_n^{-1} : n \in \omega\}$ is a left factor of some factorization of G .

Theorem 1 *If a topological group G contains an injective convergent sequence $(a_n)_{n \in \omega}$ then G is box ω -resolvable.*

Proof. We suppose that $(a_n)_{n \in \omega}$ converges to the identity e of G and denote

$$A = \{e, a_n, a_n^{-1} : n \in \omega\}, A_n = \{e, a_m, a_m^{-1} : m \leq n\}, C_n = A \setminus A_n.$$

Replacing G to the subgroup of G generated by A , in view of (1), we may suppose that G is countable, $G = \{g_n : n \in \omega\}$, $g_0 = e$. Our goal is to find a factorization $G = AB$ such that B is dense in G . We shall construct a family $\{B_n : n \in \omega\}$, $B_n \subset B_{n+1}$ of finite subsets of G such that, for each $n \in \omega$,

(3) AB_n is a partial factorization;

(4) $\{g_0, \dots, g_n\} \subset AB_n$;

(5) for every $g \in A_{n-1}B_{n-1}$, there exists $b \in B_n$ such that $g \in C_n^2b$.

After ω steps, we put $B = \bigcup_{n \in \omega} B_n$. By (3) and (4), AB is a factorization of G . By (4) and (5), B is dense in G .

We put $B_0 = \{e\}$ and suppose that we have chosen B_0, \dots, B_n satisfying (3), (4) and (5).

To make the inductive step from n to $n+1$, we use the following observation.

(6) If F is a finite subset of G and $g \notin AF$ then there is $k \in \omega$ such that $AC_kg \cap AF = \emptyset$.

Indeed, if $AC_kg \cap AF = \emptyset$ for each $k \in \omega$ then there are an injective sequence $(s_n)_{n \in \omega}$ in A and $a \in A$ such that $as_n g \in AF$ for each n , so $g \in a^{-1}F$ and $g \in AF$.

We choose the first element $g \in \{g_n : n \in \omega\} \setminus AB_n$ and use (6) with $F = B_n$ to find $k \in \omega$ such that $AC_kg \cap AB_n = \emptyset$. We take $c \in C_k$ and notice that $Acg \cap AB_n = \emptyset$ and $g \in Acg$.

We enumerate x_0, \dots, x_p the elements of the set $A_n B_n \setminus B_n$ and take $s \in C_{n+1}$ such that, for each $i \in \{0, \dots, p\}$,

$$sx_i \notin A_n B_n \bigcup Acg.$$

Then we use (6) to choose $c_0, \dots, c_p \in C_{n+1}$ such that, for each $i \in \{0, \dots, p\}$, $Ac_i(sx_i) \cap \{sx_{i+1}, \dots, sx_p\} = \emptyset$ and

$$Ac_j(sx_i) \cap (AB_n \bigcup Acg \bigcup Ac_0(sx_0) \bigcup \dots \bigcup Ac_{i-1}(sx_{i-1})) = \emptyset.$$

After that, we put

$$B_{n+1} = B_n \bigcup \{cg, c_0sx_0, \dots, c_psx_p\}$$

and note that (3), (4), (5) hold for $n+1$ in place of n . \square

Theorem 2. *Let G be an infinite totally bounded topological group of cardinality γ , H be a subgroup of G such that $|G : H| = \gamma$, F be a finite subset of G . Then the following statements hold*

(i) *there is a partial factorization FB such that B is dense in G ;*

(ii) *there is a factorization $G = HR$ such that R is dense in G ;*

In particular, G is a box n -resolvable for each $n \in \mathbb{N}$, and G is box κ -resolvable for each infinite cardinal κ , $\kappa < \gamma$.

Proof . (i) We denote $\mathfrak{F}_F = \{K \subset G : |K| < \omega, K^{-1}K \cap F^{-1}F = e\}$ and enumerate $\mathfrak{F}_F = \{K_\alpha : \alpha < \gamma\}$. We choose inductively a γ -sequence $(x_\alpha)_{\alpha < \gamma}$ in G such that

(7) $FK_\alpha^{-1}x_\alpha \cap FK_\beta^{-1}x_\beta = \emptyset$ for all $\alpha, \beta, \alpha < \beta < \gamma$, and denote $B = \bigcup_{\alpha < \gamma} K_\alpha^{-1}x_\alpha$. We take distinct $g, h \in F$. Since $K_\alpha \in \mathfrak{F}_H$, we have $gK_\alpha^{-1}x_\alpha \cap hK_\alpha^{-1}x_\alpha = \emptyset$. If $\alpha < \beta$ then, by (7), $gK_\alpha^{-1}x_\alpha \cap hK_\beta^{-1}x_\beta = \emptyset$. Hence, $gB \cap hB = \emptyset$ and the product FB is a partial factorization.

To prove that B is dense, we use

(8) for any open subsets U_1, \dots, U_n of G , there exist $y_1 \in U_1, \dots, y_n \in U_n$ such that $\{y_1, \dots, y_n\} \in \mathfrak{F}_F$, that can be easily proved by induction on n .

Now let U be a neighborhood of e and $g \in G$. We show that $Ug \cap B \neq \emptyset$. We take a neighborhood V of e such that $V^{-1}V \subseteq U$. Since G is totally bounded, there are $z_1, \dots, z_n \in G$ such that $G = z_1V \cup \dots \cup z_nV$. We use (8) to find $y_1 \in z_1V, \dots, y_n \in z_nV$ such that $\{y_1, \dots, y_n\} \in \mathfrak{F}_F$. Then $z_1 \in y_1V^{-1}, \dots, z_n \in y_nV^{-1}$ so $\{y_1, \dots, y_n\}U = G$. We chose $\alpha < \gamma$ such that $K_\alpha = \{y_1, \dots, y_n\}$. Since $K_\alpha U_g = G$, we have $x_\alpha \in K_\alpha U_g$, $K_\alpha^{-1}x_\alpha \cap Ug \neq \emptyset$ and $B \cap Ug \neq \emptyset$.

(ii) For any open subset U of G , we choose a finite subset F_U such that $G = F_U^{-1}U$ and $Hx \cap Hy = \emptyset$ for all distinct $x, y \in F_U$. We enumerate without repetitions the set $\{F_U : U \text{ is open}\}$ as $\{K_\alpha : \alpha < \gamma\}$. Since $|G : H| = \gamma$, we can choose inductively a γ -sequence $(x_\alpha)_{\alpha < \gamma}$ in G such that $HK_\alpha x_\alpha \cap HK_\beta x_\beta = \emptyset$ for all $\alpha < \beta < \gamma$. We denote $S = \bigcup_{\alpha < \gamma} K_\alpha x_\alpha$ and show that S is dense in G . Given any open subset U in G , we choose $\alpha < \gamma$ such that $G = K_\alpha^{-1}U$. Then $x_\alpha \in K_\alpha^{-1}U$ so $K_\alpha x_\alpha \cap U \neq \emptyset$ and $S \cap U \neq \emptyset$.

To conclude the proof, we complement S to some full system R of representatives of right cosets of G by H . \square

3. Comments

1. In connection with Theorem 2, we should ask

Question 1. *Is every infinite totally bounded topological group of cardinality κ box κ -resolvable?*

For $\kappa = \omega$, to answer this question positively, it suffices to generalize Theorem 1 and prove that a topological group G is box ω -resolvable provided that G contains a countable thin subset X such that e is the unique limit point of X . A subset X of G is called *thin* if $|gX \cap X| < \omega$ for every $g \in G \setminus \{e\}$. By [12, Theorem 2], every infinite totally bounded topological group G have such a subset X .

In the case $\alpha = \omega$, I believe in the positive answer to Question 1 but then

Question 2. *In ZFC, does there exist an infinite non-discrete box ω -irresolvable topological group?*

2. Given a family \mathcal{I} of subsets of a topological group G , we say that G is \mathcal{I} -box κ -resolvable if there exist a dense subset B of G and a subset A of cardinality κ such that $G = AB$ and

$aB \cap a'B \in \mathcal{I}$ for all distinct $a, a' \in A$.

Every countable totally bounded topological group G is \mathcal{I} -box ω -resolvable with respect to the family \mathcal{I} of all finite subsets of G .

By [12, Theorem 3], G has a thin dense subset B . Then $gB \cap g'B \in \mathcal{I}$ for all distinct $g, g' \in G$ and $G = GB$.

Question 3. Let τ be the topology of finite indices (see Example 1) on \mathbb{Z} . Is (\mathbb{Z}, τ) \mathcal{I} -box 2-resolvable with respect to the family \mathcal{I} of all nowhere dense subsets of \mathbb{Z} ?

3. The notion of the box resolvability is natural in much more general context of G -spaces. Let X be a topological space and let G be a discrete group. We suppose that X is endowed with transitive action $G \times X \rightarrow X : (g, x) \mapsto gx$ such that, for each $g \in G$, the mapping $x \mapsto gx$ is continuous.

We say that X is *box κ -resolvable* if there exist a dense subset B of X and a subset A of G , $|A| = \kappa$ such that $X = AB$ and the subsets $\{aB : a \in A\}$ are pairwise disjoint.

For example, take the group \mathbb{Q} of rational number with the natural topology and let G be a group of all homomorphisms of X such that, for each $g \in G$, there is $a \in \mathbb{Q}$, $a > 0$ such that $gx = x$ for every $x \in \mathbb{Q} \setminus [-a, a]$. Then \mathbb{Q} is box κ -resolvable only for $\kappa = 1$ and Theorem 1 does not hold for some G -spaces. On the other side if G is the group of all homeomorphisms of \mathbb{Q} then, by Theorem 1, \mathbb{Q} is box ω -resolvable because G contains the subgroup of translations of \mathbb{Q} .

4. A topology on a group G is called *left invariant* if all shifts $x \mapsto gx$, $g \in G$ is continuous, and a group G endowed with a left invariant topology is called *left topological*. Clearly, every left topological group has the natural structure of G -space.

In ZFC, every infinite group G admits maximal (hence, irresolvable) regular left invariant topology [10].

Every non-discrete left topological group of second category is ω -resolvable [4, Theorem 13.1.12], but in some model of ZFC there is an irresolvable homogeneous space of second category [15].

Question 4. Is every box ω -irresolvable left topological group meager?

5. We say that a left topological group is *locally box κ -resolvable* if there exists a subset B of G such that, for each neighborhood U of the identity e , we can choose $A \subseteq U$ such that $|A| = \kappa$, $e \in A$, AB is a partial factorization and the closure of each subset aB , $a \in A$ is a neighborhood of e .

If G is locally box 2-resolvable then some neighborhood of e can be partitioned into two dense subsets so G is 2-resolvable.

To see that the converse statement does not hold, we can use the semigroup structure in the Stone-Ćech compactification of a discrete group (see [7]). Given an infinite group G , we choose two idempotents p and q from $\beta G \setminus G$ such that $pq = q$, $qp = p$. We take the family $\{P \cup Q \cup \{e\} : P \in p, Q \in q\}$ as the base at e for some left invariant topology τ . Then (G, τ) is 2-resolvable but locally box 2-irresolvable. Moreover, under MA, on a countable Boolean group G , there p, q such that corresponding τ is a group topology [17].

Question 5. *In ZFC, does there exist a locally box 2-irresolvable topological group?*

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